

NOTES ON ACTIONS OF SETS AND GROUPS AND GENERALIZED AFFINE SPACES

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ABSTRACT. Some well-known and less well-known or new notions related to group actions are surveyed. Some of these notions are used to generalize affine spaces. Actions are seen as functions with values in transformation monoids.

1. INTRODUCTION

Every group action is an action of a group, but not every action of a group is a group action; for example, a group may act on a set as a semigroup. It even makes sense to say that a group acts on a set as a set; actually, it makes sense to say that a set without any given algebraic structure acts on another set [5]. These notes aim to clarify such statements, describing both well-known and less well-known or new notions related to group actions in a systematic manner. An application of independent interest, touching on differential geometry, is also presented.

Section 2 puts group actions into context. We first define actions (of sets) as a general notion and identify several properties that actions may have. Then actions of groups and some properties that may characterize such actions are defined, and group actions, monoidal actions and premonoidal actions are defined in terms of concepts already defined in Section 2. A few results relating the notions considered, selected with the application in Section 3 in mind, are also presented.

The application in question concerns generalized affine spaces. Recall that an affine space can be defined as a set equipped with a regular group action [2]. In Section 3, the notion of affine spaces is generalized to “preaffine” spaces, using the concept of premonoidal actions defined in Section 2. Section 3 also generalizes affine spaces to “multiaffine spaces”, points to a connection between “semipreaffine” and multiaffine spaces, and contains a brief discussion of measures of “non-affineness” of particular preaffine or multiaffine spaces, connecting with the notions of torsion and curvature in differential geometry. In Appendix A, we consider how to define a generalized affine space X intrinsically, using a Malcev operation on X .

2. ACTIONS

2.1. Actions of sets.

1. Let there exist a set G , a set X and a function

$$\alpha : G \rightarrow \mathcal{F}_X, \quad g \mapsto \alpha(g),$$

where \mathcal{F}_X is the set of all endofunctions on X , that is, the set of all functions $X \rightarrow X$. We call α an *action* of G on X . When α is fixed, it is convenient to write $\alpha(g)$ as \bar{g} and $\alpha(G)$ as \bar{G} . We say that \bar{G} (G) “acts on” X (through α).

The idea motivating this construction is that we intend to let G be equipped with a binary operation \square corresponding to the natural binary operation \circ on \mathcal{F}_X ,

namely function composition defined by $f \circ g(x) = f(g(x))$ or, more pedantically, $f \circ g(x) = f|_{\text{Im}(g)}(g(x))$. The simplest and strongest correspondence between \square and \circ is created by letting α be the identity map on \mathcal{F}_X and setting $\square = \circ$. More generally, we can require that $\alpha(\mathbf{g} \square \mathbf{h}) = \alpha(\mathbf{g}) \circ \alpha(\mathbf{h})$, but there are also other ways to construct a link between \square and \circ .

\mathcal{F}_X is a monoid under function composition, the so-called full transformation monoid of X ; the identity element of \mathcal{F}_X is the identity map ϵ_X . Typically, $\alpha(\mathbf{G})$ is at least a submonoid of \mathcal{F}_X , often a subgroup of \mathcal{F}_X whose elements are bijections $X \rightarrow X$, in particular automorphisms preserving some structure on X .

In this subsection and the next one, some general notions that can be defined without reference to the binary operation \square will be introduced.

2. It is clear that the equation $\bar{\mathbf{g}}(x) = y$ has a unique solution $x \in X$ for any $\bar{\mathbf{g}} \in \bar{\mathbf{G}}$ and $y \in X$ if and only if each $\bar{\mathbf{g}}$ is a bijection $X \rightarrow X$.

Now suppose that $\bar{\mathbf{g}}$ is a bijection $X \rightarrow X$. The set \mathcal{S}_X of all such bijections is a group under function composition, so there is some $\bar{\mathbf{g}}^{-1} \in \mathcal{S}_X$ such that $\bar{\mathbf{g}} \circ \bar{\mathbf{g}}^{-1} = \bar{\mathbf{g}}^{-1} \circ \bar{\mathbf{g}} = \epsilon_X$. Hence, there is some $\phi \in \mathcal{F}_X$ such that

$$(2.1) \quad \bar{\mathbf{g}}(\phi(x)) = \phi(\bar{\mathbf{g}}(x)) = x \quad \forall x \in X.$$

Note that while $\phi = \bar{\mathbf{g}}^{-1}$ is the inverse in \mathcal{S}_X of $\bar{\mathbf{g}}$, ϕ need not be the inverse in $\bar{\mathbf{G}}$ of $\bar{\mathbf{g}}$; in fact, $\bar{\mathbf{G}}$ need not even be a group. On the other hand, ϕ is the semigroup inverse in \mathcal{F}_X of $\bar{\mathbf{g}}$; we have $\phi \circ \bar{\mathbf{g}} \circ \phi = \phi$ and $\bar{\mathbf{g}} \circ \phi \circ \bar{\mathbf{g}} = \bar{\mathbf{g}}$.

Finally, if $\bar{\mathbf{g}} \in \bar{\mathbf{G}}$ is a function such that there is some $\phi \in \mathcal{F}_X$ such that (2.1) holds, then $\bar{\mathbf{g}}(x) = y$ has a solution $x = \phi(y)$ for any $y \in X$ because $\bar{\mathbf{g}}(\phi(y)) = y$, and this solution is unique because if $\bar{\mathbf{g}}(x) = y$ then $x = \phi(\bar{\mathbf{g}}(x)) = \phi(y)$.

Thus, the following conditions are equivalent:

- (1) For any $\bar{\mathbf{g}} \in \bar{\mathbf{G}}$, there is some $\phi \in \mathcal{F}_X$ such that (2.1) holds for any $x \in X$.
- (2) For any $\bar{\mathbf{g}} \in \bar{\mathbf{G}}$ and $y \in X$, $\bar{\mathbf{g}}(x) = y$ has a unique solution $x \in X$.
- (3) $\bar{\mathbf{G}}$ is a subset of \mathcal{S}_X , the set (group) of all bijections $X \rightarrow X$.

We say that α is a *reversible* action of \mathbf{G} on X if and only if one of these conditions holds, and hence all conditions hold.

3. A *unital* action (of sets) is an action α of \mathbf{G} on X such that there is some $\mathbf{g} \in \mathbf{G}$ such that $\bar{\mathbf{g}}(x) = x$ for any $x \in X$, meaning that $\epsilon_X = \bar{\mathbf{g}} \in \bar{\mathbf{G}}$.

An *invertible* action (of sets) is an action α of \mathbf{G} on X such that for every $\mathbf{g} \in \mathbf{G}$ there exists some $\mathbf{h} \in \mathbf{G}$ such that

$$(2.2) \quad \bar{\mathbf{g}}(\bar{\mathbf{h}}(x)) = \bar{\mathbf{h}}(\bar{\mathbf{g}}(x)) = x \quad \forall x \in X.$$

A comparison of (2.2) and (2.1) reveals that an invertible action is reversible. It follows from (2.2) that for every $\bar{\mathbf{g}} \in \bar{\mathbf{G}}$ there is some $\bar{\mathbf{g}}^{-1} \in \bar{\mathbf{G}}$, namely $\bar{\mathbf{h}}$, such that $\bar{\mathbf{g}}^{-1}$ is the inverse in \mathcal{S}_X (but not necessarily in $\bar{\mathbf{G}}$) of $\bar{\mathbf{g}}$. We can show as in § 2 that (2.2) holds if and only if for any $\mathbf{g} \in \mathbf{G}$ and $y \in X$ there is some $\mathbf{h} \in \mathbf{G}$ such that the equation $\bar{\mathbf{g}}(x) = y$ has the unique solution $x = \bar{\mathbf{h}}(y)$.

A *closed* action (of sets) is an action α of \mathbf{G} on X such that for any $\mathbf{g}, \mathbf{h} \in \mathbf{G}$ there is some $\mathbf{k} \in \mathbf{G}$ such that $\bar{\mathbf{g}}(\bar{\mathbf{h}}(x)) = \bar{\mathbf{k}}(x)$ for all $x \in X$, or equivalently $\bar{\mathbf{g}} \circ \bar{\mathbf{h}} \in \bar{\mathbf{G}}$.

Note that if α is closed then $\bar{\mathbf{G}}$ is a semigroup. We have $\epsilon_X \circ \bar{\mathbf{g}} = \bar{\mathbf{g}} \circ \epsilon_X = \bar{\mathbf{g}}$ for all $\bar{\mathbf{g}} \in \bar{\mathbf{G}}$, so if α is closed and unital so that $\epsilon_X \in \bar{\mathbf{G}}$ then $\bar{\mathbf{G}}$ is a monoid, and if α is closed and invertible then $\bar{\mathbf{G}}$ is a group because $\epsilon_X = \bar{\mathbf{g}} \circ \bar{\mathbf{g}}^{-1} = \bar{\mathbf{g}}^{-1} \circ \bar{\mathbf{g}} \in \bar{\mathbf{G}}$ for all $\bar{\mathbf{g}} \in \bar{\mathbf{G}}$, so that ϵ_X is the identity element in $\bar{\mathbf{G}}$ and $\bar{\mathbf{g}}^{-1}$ is the inverse in $\bar{\mathbf{G}}$ of $\bar{\mathbf{g}}$.

2.2. Duals of actions.

4. A function $f : G \rightarrow X$ defines two important set objects: a subset $f(G)$ of X and a partition of G corresponding to the equivalence relation \sim given by

$$g \sim h \iff f(g) = f(h).$$

$f^{-1}(x)$, where $x \in f(G)$, denotes the equivalence class of all $g \in G$ such that $f(g) = x$. f is surjective if and only if $f(G) = X$, and injective if and only if $f^{-1}(x)$ contains exactly one element of G for each $x \in f(G)$, so f is bijective if and only if both these conditions hold.

Let \mathcal{F}_{GX} be the set of all functions $G \rightarrow X$. Any action α of G on X has a *dual*

$$\alpha^* : X \rightarrow \mathcal{F}_{GX}$$

given by $\alpha^*(x)(g) := \alpha(g)(x)$ for all $x \in X$ and $g \in G$. Each $\alpha^*(x)$ is thus a function $G \rightarrow X$; for convenience we write $\alpha^*(x)$ as \bar{x} when α^* (or equivalently α) is fixed. The subset $\bar{x}(G)$ of X is called the *orbit* of x , and $\bar{x}^{-1}(y)$, where $x \in X$ and $y \in \bar{x}(G)$, is called the *conduit set* from x to y . This conduit set is a subset of G , namely $\{g \mid \bar{x}(g) = y\} = \{g \mid \bar{g}(x) = y\}$.

REMARK. Orbits and related concepts are usually defined in the context of group actions. In that context, it suffices to consider conduit sets of the form $\bar{x}^{-1}(x)$, each of which corresponds to the *stabilizer group* for x ; this is a subgroup of the group that acts on X . Here we formulate definitions of concepts of this kind in a context with much less structure. Actually, the situation considered here is analogous to that in § 2: there we were concerned with equations of the form $\bar{g}(x) = y$, with x unknown; here we are concerned with equations of the form $\bar{x}(g) = y$, with g unknown.

5. We shall only consider the most important concepts related to the notions in the preceding paragraph. An action α of G on X is said to be (a) *transitive*, (b) *free*, or (c) *regular* if and only if \bar{x} is (a') *surjective*, (b') *injective*, or (c') *bijective* for each $x \in X$; or equivalently, if and only if (a'') the orbit $\bar{x}(G)$ of x is X for each $x \in X$, (b'') the conduit set $\bar{x}^{-1}(y)$ contains exactly one element of G for each $x \in X$ and each $y \in \bar{x}(G)$, or (c'') conditions (a'') and (b'') hold; or equivalently, if and only if there is (a''') *at least one*, (b''') *at most one*, or (c''') *exactly one* $g \in G$ such that $\bar{g}(x) = y$ for any $x, y \in X$.

Note that if α is regular then \bar{g} , too, is uniquely determined by the requirement that $\bar{g}(x) = y$ for given $x, y \in X$.

Assuming that α is regular and denoting the unique $g \in G$ such that $\bar{g}(x) = y$ by y/x , we have

$$(2.3) \quad \overline{y/x}(x) = y,$$

$$(2.4) \quad \bar{g}(x)/x = g.$$

5†. If α is a free action and $\bar{g}(x) = \bar{h}(x)$ for some $x \in X$ and $g, h \in G$ then $g = h$. Indeed, if $\bar{g}(x) = \bar{h}(x) = y$ for some $x \in X$ but $g \neq h$ then α is not a free action. Hence, if α is a free action then the following conditions are equivalent, since obviously (2) implies (3) and (3) implies (1).

- (1) $\bar{g}(x) = \bar{h}(x)$ for some $x \in X$.
- (2) $g = h$.
- (3) $\bar{g} = \bar{h}$.

In particular, $\bar{g} = \bar{h}$ implies $g = h$, so a free action is an injective function.

2.3. Actions of groups.

6. In this subsection, we let the set \mathbf{G} be a group G with generic elements denoted g, h, \dots and identity element e . The binary operation \square in G linked to function composition \circ in \mathcal{F}_X by an action α is simply group multiplication.

A *unital action of groups* is an action α of G on X such that

$$\bar{e}(x) = x \quad \forall x \in X,$$

or $\bar{e} = \epsilon_X$. A unital action of groups is trivially unital as an action of sets.

An *invertible action of groups* is an action α of G on X such that for every $g \in G$

$$(2.5) \quad \bar{g}(\bar{g}^{-1}(x)) = \bar{g}^{-1}(\bar{g}(x)) = x \quad \forall x \in X.$$

A comparison of (2.5) and (2.2) shows that an invertible action of groups is invertible as an action of sets, and hence a reversible action. Thus, \bar{g}^{-1} is the inverse in \mathcal{S}_X of \bar{g} (§ 2), but not necessarily the inverse in \bar{G} of \bar{g} even though $\bar{g}^{-1} \in \bar{G}$. (There are trivial counterexamples for $G = \{e\}$.) Again reasoning as in § 2, we see that α is an invertible action of groups if and only if the equation $\bar{g}(x) = y$ has the unique solution $x = \bar{g}^{-1}(y)$ for every $g \in G$ and $y \in X$.

A *closed action of groups* (or *semigroup action of groups*) is an action of G on X such that

$$\bar{g}(\bar{h}(x)) = \overline{gh}(x) \quad \forall g, h \in G, \forall x \in X,$$

or $\bar{g} \circ \bar{h} = \overline{gh}$. A closed action of groups is clearly closed as an action of sets.

It is obvious what to mean by transitive, free or regular actions of groups for various kinds of actions of groups. For example, a regular unital action of groups is a unital action of groups that is regular as an action of sets.

6†. If α is a free unital action of groups and $\bar{g}(x_0) = x_0$ for some $g \in G$ and $x_0 \in X$, so that $\bar{g}(x_0) = \epsilon_X(x_0) = \bar{e}(x_0)$, then $g = e$ by § 5†, and $\bar{g} = \bar{e} = \epsilon_X$.

2.4. Monoidal and premonoidal actions of groups; translations.

7. If α is a closed action of groups then \bar{G} is a group. To verify this, first recall that $\bar{g} \circ \bar{h} = \overline{gh} \in \bar{G}$ for any $\bar{g}, \bar{h} \in \bar{G}$, so \bar{G} is a semigroup. Also, $\bar{e} \circ \bar{g} = \overline{eg} = \bar{g} = \overline{ge} = \bar{g} \circ \bar{e}$ for all $\bar{g} \in \bar{G}$, so \bar{e} is the unique identity element in \bar{G} . Finally, $\bar{g} \circ \bar{g}^{-1} = \overline{gg^{-1}} = \bar{e} = \overline{g^{-1}g} = \bar{g}^{-1} \circ \bar{g}$ for all $\bar{g} \in \bar{G}$, so \bar{g}^{-1} is the unique inverse in \bar{G} of \bar{g} . Thus, α is a group homomorphism $G \rightarrow \bar{G}$, but α need not be a group homomorphism $G \rightarrow \mathcal{S}_X$. In particular, the identity element \bar{e} in \bar{G} is not necessarily equal to the identity element ϵ_X in \mathcal{F}_X and \mathcal{S}_X .

EXAMPLE. Consider a set $X = \{a, b, c, d\}$ and the functions

$$\begin{aligned} \varepsilon &: a \mapsto a, b \mapsto a, c \mapsto c, d \mapsto d; \\ \phi &: a \mapsto a, b \mapsto a, c \mapsto d, d \mapsto c. \end{aligned}$$

One easily verifies that $\varepsilon \circ \varepsilon = \phi \circ \phi = \varepsilon$, $\varepsilon \circ \phi = \phi \circ \varepsilon = \phi$. Thus, $\Gamma = \{\varepsilon, \phi\}$ is a group under function composition with identity element ε . Consider an abstract group $G = \{e, f\}$ where $ee = ff = e$, $ef = fe = f$, so that e is the identity element in G . Let

$$\alpha : e \mapsto \varepsilon, f \mapsto \phi$$

be an action of G on X . Then α is a closed action of groups, $\alpha(G) = \Gamma$ is a group, and $\alpha(e)$ is the identity element ε in Γ . Note that ε is not a bijection. Thus, the identity element in Γ is not the identity element in the monoid \mathcal{F}_X , and also not the identity element in the group \mathcal{S}_X .

8. The previous paragraph suggests that we should look for a notion that is stronger than that of a closed action of groups. A *monoidal action* α of G on X is a *unital*, closed action of groups. Recall that \overline{G} is a group with identity element \overline{e} and that $\overline{g} \circ \overline{g^{-1}} = \overline{e} = \overline{g^{-1}} \circ \overline{g}$ for all $g \in G$ (§ 7). Hence, $\overline{e} = \epsilon_X$ implies that α is an invertible action of groups and therefore a reversible action of G on X , and that $\overline{g^{-1}}$ is the inverse in \mathcal{S}_X of \overline{g} . Thus, we have $\alpha(e) = \epsilon_X$, $\alpha(g^{-1}) = \alpha(g)^{-1}$, $\alpha(gh) = \alpha(g) \circ \alpha(h)$, and $\overline{G} \subset \mathcal{S}_X$, so α is a group homomorphism $G \rightarrow \mathcal{S}_X$, also known as a *group action* of G on X .

It turns out that a notion that is somewhat weaker than the one just defined is surprisingly useful, namely a *premonoidal action* of G on X , defined as a unital action of groups that is closed as an action of *sets*.

A *regular (pre)monoidal action* is, of course, a (pre)monoidal action that is regular as an action of sets.

9. Let ρ and σ be, respectively, a monoidal and a premonoidal action of G on X . By § 8, $\rho(G)$ is a group of transformations of X with \circ as binary operation. Although $\sigma(G)$ need not be a group, a connection between G and $\sigma(G)$ similar to that between G and $\rho(G)$ exists if σ is a *regular* premonoidal action of G on X .

To show this, first use the fact that σ is closed as an action of sets, which means that for any $g, h \in G$ we have $\overline{g} \circ \overline{h} \in \sigma(G)$, so $\sigma(G)$ is a semigroup under function composition. Also, σ is a unital action of groups, so $\epsilon_X = \overline{e} \in \sigma(G)$, and as $\epsilon_X \circ \overline{g} = \overline{g} = \overline{g} \circ \epsilon_X$ for any $\overline{g} \in \sigma(G)$, ϵ_X is the identity element in $\sigma(G)$. Finally, σ is regular, so for any $\overline{g} \in \sigma(G)$ and any $x_0 \in X$ there is a unique $\overline{h} \in \sigma(G)$ such that $\overline{h}(\overline{g}(x_0)) = x_0$, and hence $\overline{g}(\overline{h}(\overline{g}(x_0))) = \overline{g}(x_0)$. By assumption, $\overline{h} \circ \overline{g}, \overline{g} \circ \overline{h} \in \sigma(G)$, so by § 6† we have $\overline{h} \circ \overline{g} = \overline{g} \circ \overline{h} = \epsilon_X$ since σ is a free unital action of groups. Hence, $\overline{h} \in \sigma(G)$ is the inverse in $\sigma(G)$ of \overline{g} . We have thus shown that $\sigma(G)$ is a group of functions $X \rightarrow X$ under function composition.

Note that σ is a reversible action since it is invertible as an action of sets. We call the bijections in $\sigma(G)$ *translations*, and $\sigma(G)$ is accordingly called a *translation group* on X . As σ is a free action, translations are characterized by the property that ϵ_X is the only translation such that $\epsilon_X(x_0) = x_0$ for some $x_0 \in X$; any translation that has a fixed point has only fixed points.

Also note that σ is injective, since it is free (§ 5†), so σ is a bijection between G and $\sigma(G)$. If σ happens to be a regular *monoidal* action, or equivalently a regular group action, then σ is an isomorphism between G and $\sigma(G)$, but in general the bijection between G and $\sigma(G)$ is not an isomorphism. This fact is illustrated by the following example.

EXAMPLE. Cayley tables describing the cyclic group of order eight C_8 and the quaternion group Q are shown below.

C_8	e	a	a^2	a^3	a^4	a^5	a^6	a^7
e	e	a	a^2	a^3	a^4	a^5	a^6	a^7
a	a	a^2	a^3	a^4	a^5	a^6	a^7	e
a^2	a^2	a^3	a^4	a^5	a^6	a^7	e	a
a^3	a^3	a^4	a^5	a^6	a^7	e	a	a^2
a^4	a^4	a^5	a^6	a^7	e	a	a^2	a^3
a^5	a^5	a^6	a^7	e	a	a^2	a^3	a^4
a^6	a^6	a^7	e	a	a^2	a^3	a^4	a^5
a^7	a^7	e	a	a^2	a^3	a^4	a^5	a^6

Q	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

Let $\mathcal{T}_Q = \{\widehat{1}, \widehat{i}, \widehat{j}, \widehat{k}, \widehat{-1}, \widehat{-i}, \widehat{-j}, \widehat{-k}\}$ be bijections $Q \rightarrow Q$ defined by

$$\widehat{z}(x) = zx \quad \forall x, z \in Q.$$

$\tau : z \mapsto (x \mapsto zx)$ is a bijection $Q \rightarrow \mathcal{T}_Q$ because τ is clearly surjective and also injective since $zx = z'x$ implies $z = z'$. Furthermore, $\widehat{z}\widehat{y}(x) = \widehat{z} \circ \widehat{y}(x)$ for all $x, y, z \in Q$ because Q is associative, and $\widehat{1} = \epsilon_Q$. \mathcal{T}_Q is clearly a group isomorphic to Q .

Now consider the function

$$\sigma : C_8 \rightarrow \mathcal{T}_Q,$$

$$e \mapsto \widehat{1}, a \mapsto \widehat{i}, a^2 \mapsto \widehat{j}, a^3 \mapsto \widehat{k}, a^4 \mapsto \widehat{-1}, a^5 \mapsto \widehat{-k}, a^6 \mapsto \widehat{-j}, a^7 \mapsto \widehat{-i}.$$

σ is an action of groups of C_8 on Q , *unital* since $\sigma(e) = \widehat{1} = \epsilon_Q$; *closed* as an action of sets since $\sigma(a^m) \circ \sigma(a^n) = \widehat{x} \circ \widehat{y} = \widehat{xy} \in \mathcal{T}_Q = \sigma(C_8)$; and *regular* as an action of sets since the equation $zx = y$ has a unique solution $z = yx^{-1} \in Q$ for any $x, y \in Q$ so that the equation $\sigma(a^n)(x) = y$ has a unique solution $a^n = \sigma^{-1}(\widehat{yx^{-1}}) \in C_8$ for any $x, y \in Q$. Thus, σ is a regular premonoidal action of C_8 on Q , σ is a bijection $C_8 \rightarrow \mathcal{T}_Q$ by its definition, and \mathcal{T}_Q is a group. Yet, C_8 and \mathcal{T}_Q are not isomorphic, even though in this example, by design, $\sigma(g^{-1}) = \sigma(g)^{-1}$ for all $g \in C_8$. In fact, C_8 is abelian while \mathcal{T}_Q , being isomorphic to Q , is non-abelian.

2.5. Actions and binary functions.

10. For any action α of G on X there is a unique corresponding binary function

$$(2.6) \quad \beta : G \times X \rightarrow X, \quad (g, x) \mapsto gx,$$

alternatively denoted $\widehat{\alpha}$, defined by

$$(2.7) \quad gx := \overline{g}(x) \quad \forall g \in G, \forall x \in X.$$

Conversely, for any binary function β of the form (2.6) there is a unique action α , alternatively denoted $\widehat{\beta}$, defined by

$$(2.8) \quad \overline{g} := g, \quad \overline{g}(x) := gx \quad \forall g \in G, \forall x \in X.$$

We can thus write $\overline{g}(x)$ as gx , and we have the following correspondences:

α	$\widehat{\alpha}$
$\overline{y/x}(x) = y$	$(y/x)x = y$
$\overline{g}(x)/x = g$	$gx/x = g$
$\overline{g}(x) = x$	$gx = x$
$\overline{g}(\overline{h}(x)) = \overline{h}(\overline{g}(x)) = x$	$g(hx) = h(gx) = x$
$\overline{g} \circ \overline{h}(x) = \overline{g}(\overline{h}(x)) = \overline{k}(x)$	$g(hx) = kx$

The formal properties of a binary function β are similar to the formal properties of the corresponding action $\hat{\beta}$. For example, β represents a reversible action $\hat{\beta}$ if and only if the equation $gx = y$ has a unique solution x for any $g \in G$ and $y \in X$.

Note that the set of functions $G \times X \rightarrow X$ is in one-to-one correspondence with the set of functions $G \rightarrow \mathcal{F}_X$ rather than the set of functions $G \rightarrow \mathcal{X}$. This suggests that an action should indeed be defined as a function $\alpha : G \rightarrow \mathcal{F}_X$ rather than a function $\alpha : G \rightarrow \mathcal{X}$ (if actions are not defined as binary functions).

11. Let G be a group G with generic elements g, h, \dots and identity element e , so that we can write (2.6) as

$$(2.9) \quad \beta : G \times X \rightarrow X, \quad (g, x) \mapsto gx.$$

The correspondence between $\bar{g}(x)$ and gx is now rendered as a correspondence between $\bar{g}(x)$ and gx , and we can rewrite the displayed correspondences in the previous paragraph accordingly. We also have some further correspondences:

α	$\hat{\alpha}$
$\bar{e}(x) = x$	$ex = x$
$\bar{g}(\bar{g}^{-1}(x)) = \bar{g}^{-1}(\bar{g}(x)) = x$	$g(g^{-1}x) = g^{-1}(gx) = x$
$\bar{g} \circ \bar{h}(x) = \bar{g}(\bar{h}(x)) = \bar{gh}(x)$	$g(hx) = (gh)x$

In the literature, a group action is mostly defined as a binary function β such that $ex = x$ and $g(hx) = (gh)x$ for all $g, h \in G$ and all $x \in X$. A binary function of this kind represents a monoidal action $G \rightarrow \mathcal{F}_X$, where $\bar{e}(x) = x$ and $\bar{g}(\bar{h}(x)) = \bar{gh}(x)$, and such an action is invertible and hence reversible (§ 8). Alternatively, $g(g^{-1}x) = (gg^{-1})x = ex = x$ and similarly $g^{-1}(gx) = x$ for all $g \in G$ and $x \in X$, so $gx = y$ has the unique solution $x = g^{-1}y$ for any $g \in G$ and $y \in X$, implying that $\hat{\beta}$ is reversible. The assumption $ex = x$ is thus introduced to ensure that β represents a group action $\hat{\beta} : G \rightarrow \mathcal{X}$.

12. Let us briefly consider the consequences of defining an action as a binary function $G \times X \rightarrow X$ rather than a function $G \rightarrow \mathcal{F}_X$, so that $\bar{g}(x) = y$ is replaced by $gx = y$. This definition and notation encourages us to identify the element g of the group G with the corresponding function $\bar{g} : X \rightarrow X$ that sends x to y , so we say that g “acts on” x or that G “acts on” X . This identification tends to work well as long as we are dealing with a group action, because the identity $(gh)x = g(hx)$ means that we can think about the group product gh as the “product” $g \circ h$ of g and h regarded as functions. (For contravariant actions, the “product” of g and h is, analogously, $h \circ g$; see § 14.) This way of thinking may work well even if α is not injective, because we do not automatically assume that $g_1x = g_2x$ implies $g_1 = g_2$.

Identifying g and \bar{g} can lead to confusion when we do not have $\bar{gh} = \bar{g} \circ \bar{h}$ or $\bar{gh} = \bar{h} \circ \bar{g}$, however. For example, the argument in § 9 depends on the distinction between g and \bar{g} – recall, in particular, that C_8 is an abelian group while $\overline{C_8} = \mathcal{T}_Q$ is a non-abelian group. In the application in Section 3, the distinction between g and \bar{g} takes the form of a distinction between the *vector* \mathbf{v} (an algebraic concept) and the corresponding *translation* $\bar{\mathbf{v}}$ (a geometric concept). In this case, G is the abelian additive group of a vector space, while \overline{G} is a possibly non-abelian group of translations. (In particular, compare (3.5), valid in an affine space, with (3.7), valid in a preaffine space.)

If we define actions as done in § 1, we can still use binary functions $G \times X \rightarrow X$ to *represent* actions, but this is not necessary, although it may be helpful in some cases.

2.6. On notation.

13. The expression $\bar{g}(x)$ uses *left-handed notation* for actions, but we can also use *right-handed notation*, writing $(x)\bar{g}$ instead of $\bar{g}(x)$, $((x)\bar{h})\bar{g}$ or $(x)\bar{h} \circ \bar{g}$ instead of $\bar{g}(\bar{h}(x))$ or $\bar{g} \circ \bar{h}(x)$, and so on. A convenient feature of right-handed notation is that the function first applied to its argument comes first when reading from left to right.

To give expressions a smooth appearance, one may use *overloaded* notation, where function composition and function application are denoted by the same symbol. For example, one may write $\bar{g} \circ \bar{h}(x)$ as $\bar{g} \circ \bar{h} \circ x$ and $(x)\bar{h} \circ \bar{g}$ as $x \circ \bar{h} \circ \bar{g}$ – note that we do not need parentheses due to the definition of function composition.

We may specifically refer to this type of notation as *multiplicative* overloaded notation. In certain contexts, it is more intuitive to write $\bar{g} + \bar{h} + x$ or $x + \bar{h} + \bar{g}$, using *additive* overloaded notation instead of multiplicative.

We can even overload all three operators involved, writing, for example,

$$x + \bar{g} + \bar{h} = x + \overline{g + h}$$

instead of $(x)\bar{g} \circ \bar{h} = (x)\overline{gh}$, $x \circ \bar{g} \circ \bar{h} = x \circ \overline{gh}$ or $x + \bar{g} + \bar{h} = x + \overline{gh}$. We call this *totally overloaded notation*.

It is useful to have different ways of writing y/x (§ 5) in left-handed and right-handed notation. We may write y/x as $^y/_x$ when using left-handed multiplicative notation and as $x \setminus ^y$ when using right-handed multiplicative notation; (2.3) then becomes $^y/_x \circ x = y$ and $x \circ x \setminus ^y = y$, respectively.

The expression corresponding to y/x in additive notation is $y - x$. We may write this as $(y \leftarrow x)$ when using left-handed notation and as $(x \rightarrow y)$ when using right-handed notation. Relation (2.3) becomes $\overline{(y \leftarrow x)} + x = y$ and $x + \overline{(x \rightarrow y)} = y$, respectively; both these expressions convey the idea that x is moved to y by some transformation on X .

2.7. Covariant and contravariant actions of groups.

14. Closed actions of groups as defined above are *covariant* in the sense that $\overline{gh}(x) = \bar{g} \circ \bar{h}(x)$, but it is also possible to define *contravariant* actions such that $\overline{gh}(x) = \bar{h} \circ \bar{g}(x)$. In right-handed notation, we have $(x)\overline{gh} = (x)\bar{h} \circ \bar{g}$ for a covariant action, $(x)\overline{gh} = (x)\bar{g} \circ \bar{h}$ for a contravariant action.

We note that covariant actions look most natural in left-handed notation, whereas contravariant actions look most natural in right-handed notation. As a consequence, covariant actions are traditionally called “left actions”, while contravariant actions are called “right actions”. Although the present terminology breaks with this tradition, it has the virtue of clearly separating notation from concepts. (Furthermore, I prefer the terminology covariant – contravariant to alternatives such as “homomorphic” – “antihomomorphic”.)

It should be kept in mind, though, that covariant and contravariant actions are exchangeable on the conceptual level. We can transform a covariant (contravariant) action α of G on X to a contravariant (covariant) action α^{op} of G on X , defined by $\alpha^{\text{op}}(gh) := \alpha(hg)$ for all $g, h \in G$.

3. SKETCH OF AN APPLICATION:
AFFINE, (SEMI)PREAFFINE AND MULTIAFFINE SPACES

3.1. Affine and preaffine spaces.

15. In this section, V will denote a vector space over some field K , with generic elements $\mathbf{u}, \mathbf{v}, \dots$ and identity element $\mathbf{0}$. An *action* of V on a set X is an action of the additive group of V on X . We define an *affine space* as a non-empty set X equipped with a *regular monoidal* action α of V on X , or equivalently (see § 8) a regular group action of V on X [2].

We shall use *right-handed, totally overloaded additive notation* in this section, so we write $\mathbf{u} \square \mathbf{v}$ as $\mathbf{u} + \mathbf{v}$, $\bar{\mathbf{v}}(x)$ as $x + \bar{\mathbf{v}}$, and $\bar{\mathbf{v}}(\bar{\mathbf{u}}(x))$ as $x + \bar{\mathbf{u}} + \bar{\mathbf{v}}$. Closed actions of groups are assumed to be *contravariant* so that $\overline{\mathbf{u} + \mathbf{v}}(x) = \bar{\mathbf{v}}(\bar{\mathbf{u}}(x))$, written as $x + \overline{\mathbf{u} + \mathbf{v}} = x + \bar{\mathbf{u}} + \bar{\mathbf{v}}$.

By the definition of an affine space, α is a unital, closed action of groups, so

$$(3.1) \quad x + \bar{\mathbf{0}} = x \quad \forall x \in X,$$

$$(3.2) \quad \bar{\mathbf{u}} + \bar{\mathbf{v}} = \overline{\mathbf{u} + \mathbf{v}} \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

Furthermore, α is a regular action, so there is a function

$$X \times X \rightarrow V, \quad (x, y) \mapsto (x \rightarrow y)$$

such that

$$(3.3) \quad x + \overline{(x \rightarrow y)} = y \quad \forall x, y \in X,$$

$$(3.4) \quad (x \rightarrow (x + \bar{\mathbf{v}})) = \mathbf{v} \quad \forall x \in X, \forall \mathbf{v} \in V,$$

corresponding to (2.3) and (2.4), respectively. By (3.3), $\mathbf{v} = (x \rightarrow y)$ is a solution of the equation $x + \bar{\mathbf{v}} = y$, and (3.4) ensures that this solution is unique, because if $x + \bar{\mathbf{u}} = x + \bar{\mathbf{v}}$ then $(x \rightarrow (x + \bar{\mathbf{u}})) = (x \rightarrow (x + \bar{\mathbf{v}}))$, so $\mathbf{u} = \mathbf{v}$.

The known properties of affine spaces can be derived from these assumptions. For example, “Chasles’ law”

$$(3.5) \quad (x \rightarrow y) + (y \rightarrow z) = (x \rightarrow z) \quad \forall x, y, z \in X,$$

can be derived by noting that (3.2) and (3.3) imply

$$x + \overline{(x \rightarrow y) + (y \rightarrow z)} = x + \overline{(x \rightarrow y)} + \overline{(y \rightarrow z)} = z,$$

and (3.4) thus implies

$$(x \rightarrow z) = \left(x \rightarrow \left(x + \overline{(x \rightarrow y) + (y \rightarrow z)} \right) \right) = (x \rightarrow y) + (y \rightarrow z).$$

It is worth noting that

$$(3.6) \quad x + \bar{\mathbf{u}} + \bar{\mathbf{v}} = x + \overline{\mathbf{u} + \mathbf{v}} = x + \overline{\mathbf{v} + \mathbf{u}} = x + \bar{\mathbf{v}} + \bar{\mathbf{u}} \quad \forall x \in X, \forall \mathbf{u}, \mathbf{v} \in V.$$

Geometrically, this is the so-called parallelogram law of vector addition in an affine space; see Figure 3.1.

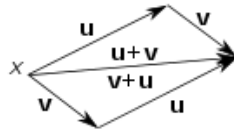


FIGURE 3.1. Composition of translations in an affine space.

16. We define a *preaffine space* as a non-empty set X equipped with a *regular premonoidal* action α of V on X . Identities (3.1), (3.3) and (3.4) are thus still required to hold, but instead of (3.2) we require that α is closed as an action of sets (§ 3). That is, we assume that

$$(3.7) \quad \overline{(x \rightarrow y)} + \overline{(y \rightarrow z)} = \overline{(x \rightarrow z)} \quad \forall x, y, z \in X.$$

In other words, the transformation composed of the translation that sends x to y followed by the translation that sends y to z is itself a translation, necessarily the one that sends x to z .

It is clear that an affine space is a preaffine space, since (3.7) follows from (3.2) and Chasles' law (3.5). A *strictly preaffine* space is a preaffine space that is not an affine space.

17. In a strictly preaffine space, we do not have $\bar{\mathbf{u}} + \bar{\mathbf{v}} = \overline{\mathbf{u} + \mathbf{v}}$ for all $\mathbf{u}, \mathbf{v} \in V$, and thus the parallelogram law (3.6) does not hold, and translations do not always commute; see Figure 3.2.



FIGURE 3.2. Commutative and non-commutative composition of translations in a non-affine space.

On the other hand, recall from § 9 that the translations $\bar{\mathbf{v}}$ form a group, albeit not necessarily an abelian group. The unique identity element in this group is $\bar{\mathbf{0}}$, and we denote the unique inverse of $\bar{\mathbf{v}}$ by $-\bar{\mathbf{v}}$, writing $\bar{\mathbf{u}} + (-\bar{\mathbf{v}})$ as $\bar{\mathbf{u}} - \bar{\mathbf{v}}$. We have

$$x + \overline{(x \rightarrow x)} = x = x + \bar{\mathbf{0}} \quad \forall x \in X,$$

and by § 5† this implies $\overline{(x \rightarrow x)} = \bar{\mathbf{0}}$ and $\overline{(x \rightarrow x)} = \bar{\mathbf{0}}$ for every $x \in X$. Hence, $\overline{(x \rightarrow y)} + \overline{(y \rightarrow x)} = \bar{\mathbf{0}}$ by (3.7), so $\overline{(x \rightarrow y)} = \overline{(x \rightarrow y)} + \overline{(y \rightarrow x)} - \overline{(y \rightarrow x)} = -\overline{(y \rightarrow x)}$.

3.2. Action fields, multiaffine spaces and associated semipreaffine spaces.

18. Recall that $x + \bar{\mathbf{g}}$ is another way of writing $\alpha(\mathbf{g})(x)$. Let us return to the latter notation temporarily while introducing a generalization of the concept of action. An *action field* of \mathbf{G} on X is a function Φ that to every $p \in X$ assigns an action

$$\Phi(p) : \mathbf{G} \rightarrow \mathcal{F}_X$$

of \mathbf{G} on X in such a way that $\Phi(p)(\mathbf{g}) = \Phi(q)(\mathbf{g}) =: \bar{\mathbf{g}}$ for any $p, q \in X$. When dealing with an action field, we thus have expressions of the form $\Phi(p)(\mathbf{g})(x)$ instead of $\alpha(\mathbf{g})(x)$, or $\Phi(p)(\mathbf{v})(x)$ instead of $\alpha(\mathbf{v})(x)$ if \mathbf{G} is the additive group of a vector space V . In the latter case we of course write \bar{V} instead of $\bar{\mathbf{G}}$.

In right-handed additive notation, we write $\Phi(p)(\mathbf{v})(x)$ as

$$(3.8) \quad x + \bar{\mathbf{v}}^p.$$

Analogously, we denote the unique solution \mathbf{v} of the equation $x + \bar{\mathbf{v}}^p = y$ by

$$(3.9) \quad (x \rightarrow y)_p.$$

For any $p \in X$, $\overline{(x \rightarrow y)_p}^p$ is thus the unique translation that sends x to y .

19. A *regular monoidal action field* is an action field Φ of V on X such that $\Phi(p)$ is a regular monoidal action of V on X for each $p \in X$. We call a non-empty set X equipped with a regular monoidal action field Φ a *pointwise affine* or *multiaffine* space, since X is an affine space with respect to $\Phi(p)$ for each $p \in X$. (A *strictly multiaffine* space is a multiaffine space with a non-constant action field.) Specifically, a multiaffine space is characterized by the following identities:

$$(3.10) \quad x + \bar{\mathbf{0}}^p = x \quad \forall p, x \in X,$$

$$(3.11) \quad x + \bar{\mathbf{u}}^p + \bar{\mathbf{v}}^p = x + \overline{\mathbf{u} + \mathbf{v}}^p \quad \forall p, x \in X, \forall \mathbf{u}, \mathbf{v} \in V,$$

$$(3.12) \quad x + \overline{(x \rightarrow y)}_p^p = y \quad \forall p, x, y \in X,$$

$$(3.13) \quad (x \rightarrow (x + \bar{\mathbf{v}}^p))_p = \mathbf{v} \quad \forall p, x \in X, \forall \mathbf{v} \in V.$$

A *multipreaffine* space can be defined similarly as a non-empty set X equipped with a regular premonoidal action of V on X for each $p \in X$, but such spaces will not be considered here.

20. If X is a multiaffine space then each $\Phi(p)$ is a free action, so by § 5† there are for any $p \in X$ two bijective functions,

$$\begin{aligned} \phi(p) : V &\rightarrow \bar{V}, & \mathbf{v} &\mapsto \phi(p)(\mathbf{v}) = \bar{\mathbf{v}}^p, \\ \phi(p)^{-1} : \bar{V} &\rightarrow V, & \bar{\mathbf{v}}^q &\mapsto \phi(p)^{-1}(\bar{\mathbf{v}}^q) =: (\bar{\mathbf{v}}^q)_p \end{aligned}$$

such that $\phi(p)^{-1} \circ \phi(p) = \text{Id}_V$ and $\phi(p) \circ \phi(p)^{-1} = \text{Id}_{\bar{V}}$ for each $p \in X$. We also have $\phi(q) \circ \phi(q)^{-1} \circ \phi(p) = \phi(p)$ for any $p, q \in X$, and we can write

$$(3.14) \quad (\bar{\mathbf{v}}^p)_p = \mathbf{v} \quad \forall p \in X, \forall \mathbf{v} \in V,$$

$$(3.15) \quad \overline{(\bar{\mathbf{v}}^p)_q}^q = \bar{\mathbf{v}}^p \quad \forall p, q \in X, \forall \mathbf{v} \in V.$$

As $\overline{(x \rightarrow y)_p}^p = \overline{(x \rightarrow y)_q}^q$ for all $p, q \in X$, (3.14) gives

$$(3.16) \quad \left(\overline{(x \rightarrow y)_p}^p \right)_q = (x \rightarrow y)_q \quad \forall x, y, p, q \in X.$$

21. Given a multiaffine space with action field Φ of V on X , we can define a single action α_Φ of V on X by setting $\alpha_\Phi(\mathbf{v})(x) = \Phi(x)(\mathbf{v})(x)$ for all $x \in X$ and $\mathbf{v} \in V$, or in the right-handed additive notation used here,

$$(3.17) \quad x + \bar{\mathbf{v}} := x + \bar{\mathbf{v}}^x \quad \forall x \in X, \forall \mathbf{v} \in V.$$

For consistency, we define

$$(3.18) \quad (x \rightarrow y) := (x \rightarrow y)_x \quad \forall x, y \in X.$$

Substituting x for p in (3.10), (3.12) and (3.13) and using (3.17) and (3.18), we obtain (3.1), (3.3) and (3.4), respectively. On the other hand, to obtain

$$p + \overline{(x \rightarrow y)} + \overline{(y \rightarrow z)} = p + \overline{(x \rightarrow z)} \quad \forall p, x, y, z \in X$$

by using (3.17) and (3.18) we need to start from

$$p + \overline{(x \rightarrow y)_x}^p + \overline{(y \rightarrow z)_y}^{p + \overline{(x \rightarrow y)_x}^p} = p + \overline{(x \rightarrow z)_x}^p \quad \forall p, x, y, z \in X,$$

but in general the latter relation holds only if $p = x$ (by 3.12). This means that we cannot derive (3.7) from (3.10)–(3.13) by means of (3.17) and (3.18), and *a fortiori* (3.2) does not hold. We say that α_Φ defines a *semipreaffine space*.

3.3. Deformation of affine spaces.

22. It is helpful to think of a non-affine space as an affine space which has suffered some deformation. What separates an affine space from a strictly preaffine space is whether or not the identity $\overline{\mathbf{u}} + \overline{\mathbf{v}} = \overline{\mathbf{u} + \mathbf{v}}$ holds, and in this section we will consider measures of “non-affineness” based on “discrepancies” such as that between $\overline{\mathbf{u}} + \overline{\mathbf{v}}$ and $\overline{\mathbf{u} + \mathbf{v}}$. We shall relate such discrepancies to points in the space considered.

Let us first look at measures of the deformation at x of an affine space X , regarded as a non-strictly preaffine space, into a strictly preaffine space. It is natural to start with the function $\mathfrak{T}_0 : X \times V \times V \rightarrow V$, defined by the equation

$$(3.19) \quad x + \overline{\mathbf{u}} + \overline{\mathbf{v}} + \overline{\mathfrak{T}_0(x, \mathbf{u}, \mathbf{v})} = x + \overline{\mathbf{u} + \mathbf{v}} \quad \forall x \in X, \forall \mathbf{u}, \mathbf{v} \in V$$

with the unique solution

$$(3.20) \quad \mathfrak{T}_0(x, \mathbf{u}, \mathbf{v}) = ((x + \overline{\mathbf{u}} + \overline{\mathbf{v}}) \rightarrow (x + \overline{\mathbf{u} + \mathbf{v}})).$$

\mathfrak{T}_0 is a measure of “general non-affineness”; by definition X is an affine space if and only if $\mathfrak{T}_0(x, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and $\mathbf{u}, \mathbf{v} \in V$. It is obvious that $\mathfrak{T}_0(x, \mathbf{0}, \mathbf{v}) = \mathfrak{T}_0(x, \mathbf{u}, \mathbf{0}) = \mathbf{0}$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$.

23. We shall now consider a special kind of deformation of affine spaces. Modifying the definition in the previous paragraph, one can construct a translation-skewsymmetric function $\mathfrak{T}_1 : X \times V \times V \rightarrow V$ such that $\overline{\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v})} = -\overline{\mathfrak{T}_1(x, \mathbf{v}, \mathbf{u})}$ by requiring that

$$(3.21) \quad x + \overline{\mathbf{u}} + \overline{\mathbf{v}} + \overline{\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v})} = x + \overline{\mathbf{v}} + \overline{\mathbf{u}} \quad \forall x \in X, \forall \mathbf{u}, \mathbf{v} \in V,$$

so that

$$(3.22) \quad \mathfrak{T}_1(x, \mathbf{u}, \mathbf{v}) = ((x + \overline{\mathbf{u}} + \overline{\mathbf{v}}) \rightarrow (x + \overline{\mathbf{v}} + \overline{\mathbf{u}})).$$

As $(x \rightarrow x) = \mathbf{0}$, $\mathfrak{T}_1(x, \mathbf{0}, \mathbf{v}) = \mathfrak{T}_1(x, \mathbf{u}, \mathbf{0}) = \mathbf{0}$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$. Also, by (3.22) and (3.7), $\overline{\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v})} + \overline{\mathfrak{T}_1(x, \mathbf{v}, \mathbf{u})} = \overline{((x + \overline{\mathbf{u}} + \overline{\mathbf{v}}) \rightarrow (x + \overline{\mathbf{u}} + \overline{\mathbf{v}}))} = \mathbf{0}$, so $\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v}) = -\mathfrak{T}_1(x, \mathbf{v}, \mathbf{u})$ as promised. Furthermore,

$$\begin{aligned} \overline{\mathfrak{T}_0(x, \mathbf{u}, \mathbf{v})} - \overline{\mathfrak{T}_0(x, \mathbf{v}, \mathbf{u})} &= \overline{((x + \overline{\mathbf{u}} + \overline{\mathbf{v}}) \rightarrow (x + \overline{\mathbf{u} + \mathbf{v}}))} \\ &+ \overline{((x + \overline{\mathbf{v}} + \overline{\mathbf{u}}) \rightarrow (x + \overline{\mathbf{v} + \mathbf{u}}))} = \overline{((x + \overline{\mathbf{u}} + \overline{\mathbf{v}}) \rightarrow (x + \overline{\mathbf{v}} + \overline{\mathbf{u}}))}, \end{aligned}$$

so $\overline{\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v})} = \overline{\mathfrak{T}_0(x, \mathbf{u}, \mathbf{v})} - \overline{\mathfrak{T}_0(x, \mathbf{v}, \mathbf{u})}$.

This relation between \mathfrak{T}_1 and \mathfrak{T}_0 shows that if $\mathfrak{T}_0(x, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$ then $\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$. It is not true, however, that $\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ implies $\mathfrak{T}_0(x, \mathbf{u}, \mathbf{v}) = \mathbf{0}$. We say, informally at this point, that \mathfrak{T}_1 measures the “torsion” of X , and that if $\mathfrak{T}_1(x, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$ then X is “torsion-free”. Thus, an affine space is torsion-free, but a torsion-free space is not necessarily affine, as illustrated in Figure 3.2.

24. Let now X be a multiaffine space associated with a semipreaffine space. Instead of (3.21), we have for the multiaffine space

$$(3.23) \quad x + \overline{\mathbf{u}}^x + \overline{\mathbf{v}}^y + \overline{\mathfrak{T}_1^*(x, \mathbf{u}, \mathbf{v})}^t = x + \overline{\mathbf{v}}^x + \overline{\mathbf{u}}^z \quad \forall x \in X, \forall \mathbf{u}, \mathbf{v} \in V,$$

where $x + \overline{\mathbf{u}}^x = y$, $y + \overline{\mathbf{v}}^y = t$, and $x + \overline{\mathbf{v}}^x = z$, so

$$(3.24) \quad \mathfrak{T}_1^*(x, \mathbf{u}, \mathbf{v}) = ((x + \overline{\mathbf{u}}^x + \overline{\mathbf{v}}^y) \rightarrow (x + \overline{\mathbf{v}}^x + \overline{\mathbf{u}}^z))_t.$$

Corresponding to properties of \mathfrak{T}_1 , we have $\mathfrak{T}_1^*(x, \mathbf{0}, \mathbf{v}) = \mathfrak{T}_1^*(x, \mathbf{u}, \mathbf{0}) = \mathbf{0}$ and $\overline{\mathfrak{T}_1^*(x, \mathbf{u}, \mathbf{v})}^t = -\overline{\mathfrak{T}_1^*(x, \mathbf{v}, \mathbf{u})}^t$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$.

If X has a constant action field, meaning that X can be regarded as an affine space, then $\bar{\mathbf{v}}^y = \bar{\mathbf{v}}^x$ and $\bar{\mathbf{u}}^z = \bar{\mathbf{u}}^x$, so $x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^y = x + \overline{\mathbf{u} + \mathbf{v}}^x = x + \overline{\mathbf{v} + \mathbf{u}}^x = x + \bar{\mathbf{v}}^x + \bar{\mathbf{u}}^z$, corresponding to (3.6), so that $\mathfrak{T}_1^*(x, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$. This shows again that an affine space is torsion-free.

25. Let us continue with multiaffine spaces, defining y and z by $x + \bar{\mathbf{u}}^x = y$ and $y + \bar{\mathbf{v}}^y = z$. Corresponding to (3.5), we have $(x \rightarrow y)_p + (y \rightarrow z)_p = (x \rightarrow z)_p$ for each $p \in X$, and this implies

$$(3.25) \quad x + \overline{(x \rightarrow y)_x}^x + \overline{(y \rightarrow z)_y}^y = x + \overline{(x \rightarrow y)_x + (y \rightarrow z)_x}^x \quad \forall x, y, z \in X.$$

We have $(x \rightarrow y)_x = \mathbf{u}$, $(y \rightarrow z)_y = \mathbf{v}$ and also $\left(\overline{(y \rightarrow z)_y}^y\right)_x = (y \rightarrow z)_x$ by (3.16), so we can rewrite (3.25) as

$$(3.26) \quad x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^y = x + \overline{\mathbf{u} + (\bar{\mathbf{v}}^y)_x}^x \quad \forall x \in X, \forall \mathbf{u}, \mathbf{v} \in V.$$

We can use (3.26) to obtain another measure of deformation of affine spaces. Specifically, we inject $\bar{\mathbf{w}}^x$ into each side of the relation (3.26), obtaining

$$(3.27) \quad x + \bar{\mathbf{w}}^x + \bar{\mathbf{u}}^r + \bar{\mathbf{v}}^s = x + \bar{\mathbf{w}}^x + \overline{\mathbf{u} + (\bar{\mathbf{v}}^s)_r}^r, \quad \forall x \in X, \forall \mathbf{w}, \mathbf{u}, \mathbf{v} \in V,$$

where $x + \bar{\mathbf{w}}^x = r$ and $r + \bar{\mathbf{u}}^r = s$; we also set $s + \bar{\mathbf{v}}^s = t$. Note that (3.27) is a formal relation that does not necessarily hold; we can define a function $\mathfrak{C}_0^* : X \times V \times V \times V \rightarrow V$ by the equation

$$(3.28) \quad x + \bar{\mathbf{w}}^x + \bar{\mathbf{u}}^r + \bar{\mathbf{v}}^s + \overline{\mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{u}, \mathbf{v})}^t = x + \bar{\mathbf{w}}^x + \overline{\mathbf{u} + (\bar{\mathbf{v}}^s)_r}^r$$

with the unique solution

$$(3.29) \quad \mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{u}, \mathbf{v}) = \left((x + \bar{\mathbf{w}}^x + \bar{\mathbf{u}}^r + \bar{\mathbf{v}}^s) \rightarrow \left(x + \bar{\mathbf{w}}^x + \overline{\mathbf{u} + (\bar{\mathbf{v}}^s)_r}^r \right) \right)_t.$$

Clearly, $\mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{u}, \mathbf{0}) = \mathbf{0}$ for all $x \in X$ and $\mathbf{w}, \mathbf{u} \in V$, and by (3.15) $\overline{(\bar{\mathbf{v}}^s)_r}^r = \bar{\mathbf{v}}^s$, so $\mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{0}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and $\mathbf{w}, \mathbf{v} \in V$. Finally, if $\mathbf{w} = \mathbf{0}$ then (3.27) reduces to (3.26), so $\mathfrak{C}_0^*(x, \mathbf{0}, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and $\mathbf{u}, \mathbf{v} \in V$.

If X is a multiaffine space with a constant vector field so that X can be regarded as an affine space, we have $(\bar{\mathbf{v}}^s)_r = (\bar{\mathbf{v}}^r)_r = \mathbf{v}$ by (3.14). Adding to this the facts that $\bar{\mathbf{u}}^r = \bar{\mathbf{u}}^x$, $\bar{\mathbf{v}}^s = \bar{\mathbf{v}}^x$, and $\overline{\mathbf{u} + \mathbf{v}}^r = \overline{\mathbf{u} + \mathbf{v}}^x = \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^x$, we conclude that (3.27) holds. Thus, in an affine space $\mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and $\mathbf{w}, \mathbf{u}, \mathbf{v} \in V$.

Without clear justification at the moment, we say that \mathfrak{C}_0^* is a measure of “curvature”, and X is said to be “flat” if and only if $\mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{u}, \mathbf{v}) = \mathbf{0}$ for all $x \in X$ and $\mathbf{w}, \mathbf{u}, \mathbf{v} \in V$. Thus, an affine space is flat, but (3.27) does not imply

$$x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^{x+\bar{\mathbf{u}}^x} = x + \overline{\mathbf{u} + \mathbf{v}}^x \quad \forall x \in X, \forall \mathbf{u}, \mathbf{v} \in V,$$

so a flat space is not necessarily affine. (Based on the displayed relation, we could have defined a function \mathfrak{T}_0^* , a measure of “general non-affineness” analogous to \mathfrak{T}_0 .)

26. One obtains a function $\mathfrak{C}_1^* : X \times V \times V \times V \rightarrow V$ corresponding to \mathfrak{C}_0^* and translation-skewsymmetric in \mathbf{u} and \mathbf{v} by setting

$$(3.30) \quad \overline{\mathfrak{C}_1^*(x, \mathbf{w}, \mathbf{u}, \mathbf{v})} = \overline{\mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{u}, \mathbf{v})} - \overline{\mathfrak{C}_0^*(x, \mathbf{w}, \mathbf{v}, \mathbf{u})}.$$

Reasoning as in § 23, we obtain $\overline{\mathfrak{C}_1^*(x, \mathbf{w}, \mathbf{u}, \mathbf{v})} = -\overline{\mathfrak{C}_1^*(x, \mathbf{w}, \mathbf{v}, \mathbf{u})}$. We also have $\mathfrak{C}_1^*(x, \mathbf{0}, \mathbf{u}, \mathbf{v}) = \mathfrak{C}_1^*(x, \mathbf{w}, \mathbf{0}, \mathbf{v}) = \mathfrak{C}_1^*(x, \mathbf{w}, \mathbf{u}, \mathbf{0}) = \mathbf{0}$ for all $x \in X$ and all $\mathbf{w}, \mathbf{u}, \mathbf{v} \in V$.

3.4. Bound vectors and parallel transport.

27. The measures of deformation of an affine space derived in the previous subsection can be interpreted geometrically in terms of bound vectors and parallel transport.

A *bound vector* is defined for present purposes simply as a pair of points (x, y) , intuitively corresponding to a directed line segment connecting x to y . We say that x is the *origin* and y the *tip* of (x, y) .

Parallel transport of a bound vector translates its origin and tip by means of the same vector. This is often described by speaking about parallel transport of a bound vector *along* another bound vector with the same origin.

In a preaffine space, parallel transport by \mathbf{v} , or along $(x, x + \bar{\mathbf{v}})$, maps $(x, x + \bar{\mathbf{u}})$ to $(x + \bar{\mathbf{v}}, x + \bar{\mathbf{u}} + \bar{\mathbf{v}})$. Alternatively, parallel transport along (x, y) maps (x, z) to $(x + \overline{(x \rightarrow y)}, z + \overline{(x \rightarrow y)}) = (y, z + \overline{(x \rightarrow y)})$.

In a multiaffine space, parallel transport along $(x, x + \bar{\mathbf{v}}^x)$ maps $(x, x + \bar{\mathbf{u}}^x)$ to $(x + \bar{\mathbf{v}}^x, x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^y)$, where $y = x + \bar{\mathbf{u}}^x$. Alternatively, parallel transport along (x, y) maps (x, z) to $(x + \overline{(x \rightarrow y)}^x, z + \overline{(x \rightarrow y)}^y) = (y, z + \overline{(x \rightarrow y)}^y)$.

28. Let the points $x, y, z \in X$ be given. Map (x, z) to (y, t) by parallel transport along (x, y) , and map (x, y) to (z, t') by parallel transport along (x, z) . \mathfrak{T}_1 , or equivalently \mathfrak{T}_1^* , is a measure of the discrepancy between the tips $t = x + \bar{\mathbf{u}} + \bar{\mathbf{v}}$ (or $t = x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^y$) and $t' = x + \bar{\mathbf{v}} + \bar{\mathbf{u}}$ (or $t' = x + \bar{\mathbf{v}}^x + \bar{\mathbf{u}}^z$) of the two parallel-transported bound vectors (y, t) and (z, t') . Figure 3.3 visualizes this parallel-transport construction.

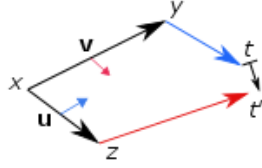


FIGURE 3.3. Geometrical interpretation of \mathfrak{T}_1 or \mathfrak{T}_1^* .

29. Let the points $x, y, z, r \in X$ be given. Map (x, r) to (y, s) by parallel transport along (x, y) , and then map (y, s) to (z, t) by parallel transport along (y, z) . Also map (x, r) to (z, t') by parallel transport along (x, z) . \mathfrak{C}_0^* is a measure of the discrepancy between the tips $t = x + \bar{\mathbf{w}}^x + \bar{\mathbf{u}}^r + \bar{\mathbf{v}}^s$ and $t' = x + \bar{\mathbf{w}}^x + \bar{\mathbf{u}} + \overline{(\bar{\mathbf{v}}^s)}^r$ of the parallel-transported bound vectors (z, t) and (z, t') , where $z = x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^y = x + \bar{\mathbf{u}} + \overline{(\bar{\mathbf{v}}^y)}^x$. Figure 3.4 shows this parallel-transport construction.

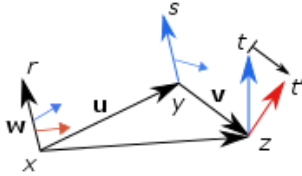


FIGURE 3.4. Geometrical interpretation of \mathfrak{C}_0^* .

3.5. Final remarks.

30. Readers may have suspected that what is presented here as an application of a theory was historically a problem that helped to inspire the construction of that theory. It is clear – for example, from synthetic differential geometry [4] – that the measures \mathfrak{T}_1 and \mathfrak{T}_1^* are related to the torsion tensor in differential geometry, while \mathfrak{C}_0^* and \mathfrak{C}_1^* are related to the curvature tensor. Note, though, that the present notions of torsion and curvature are independent of the “differential” part of differential geometry – it is possible to explain and understand the general idea of parallel transport, torsion and curvature without first introducing infinitesimal notions (of various orders). These notions can be introduced at the next stage instead, generalizing generalized affine spaces.

The present approach to “finitary differential geometry” not only opens a fast track to torsion and curvature but also suggests a path from coordinate-free affine geometry defined in terms of group actions to coordinate-free differential geometry. Recall that a *pointwise affine* space may yet be *globally non-affine*. In other words, $x + \bar{\mathbf{u}}^p + \bar{\mathbf{v}}^p = x + \overline{\mathbf{u} + \mathbf{v}}^p$ for all $p, x \in X$ and $\mathbf{u}, \mathbf{v} \in V$ does not imply $x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^{(x+\bar{\mathbf{u}}^x)} = x + \overline{\mathbf{u} + \mathbf{v}}^x$ for all $x \in X$ and $\mathbf{u}, \mathbf{v} \in V$. In differential geometry, on the other hand, we typically deal with a space that is (1) *pointwise* and (2) *infinitesimally* affine but (3) *locally* and (4) *globally* non-affine. That is,

- (1) $x + \bar{\mathbf{u}}^p + \bar{\mathbf{v}}^p = x + \overline{\mathbf{u} + \mathbf{v}}^p$ for all $p, x \in X$ and $\mathbf{u}, \mathbf{v} \in V$;
- (2) $x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^{(x+\bar{\mathbf{u}}^x)} = x + \overline{\mathbf{u} + \mathbf{v}}^x$ for all $x \in X$ and all “infinitesimal” $\mathbf{u}, \mathbf{v} \in V - X$ is “infinitesimally affine”;
- (3) we do not have $x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^{(x+\bar{\mathbf{u}}^x)} = x + \overline{\mathbf{u} + \mathbf{v}}^x$ for all $x \in X$ and all “nearly infinitesimal” $\mathbf{u}, \mathbf{v} \in V$, but \mathfrak{T}_0^* , given by

$$\mathfrak{T}_0^*(x, \mathbf{u}, \mathbf{v}) = \left(x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^{(x+\bar{\mathbf{u}}^x)} \rightarrow x + \overline{\mathbf{u} + \mathbf{v}}^x \right)_{x+\bar{\mathbf{u}}^x+\bar{\mathbf{v}}^{(x+\bar{\mathbf{u}}^x)}},$$

is a bilinear function of its “nearly infinitesimal” vector arguments – X is “locally linearly connected”;

- (4) we do not have $x + \bar{\mathbf{u}}^x + \bar{\mathbf{v}}^{(x+\bar{\mathbf{u}}^x)} = x + \overline{\mathbf{u} + \mathbf{v}}^x$ for all $x \in X$ and all $\mathbf{u}, \mathbf{v} \in V$.

It remains to give formal definitions of notions such as “infinitesimal” and “nearly infinitesimal”; this is only a heuristic description. Note, though, that the values of \mathfrak{T}_0 , \mathfrak{T}_0^* , \mathfrak{T}_1 , \mathfrak{T}_1^* , \mathfrak{C}_0^* , and \mathfrak{C}_1^* are all equal to $\mathbf{0}$ when one of their vector arguments is $\mathbf{0}$, as required if these functions are to be approximated by linear functions locally. In addition, \mathfrak{D}^* , defined by

$$x + \bar{\mathbf{v}}^p + \overline{\mathfrak{D}^*(x, p, \mathbf{d}, \mathbf{v})}^p = x + \bar{\mathbf{v}}^{p+\mathbf{d}^p},$$

satisfies $\mathfrak{D}^*(x, p, \mathbf{0}, \mathbf{v}) = \mathbf{0}$ for all $x, p \in X$ and $\mathbf{v} \in V$, establishing a necessary condition for \mathfrak{D}^* to be approximated locally (for $\mathbf{d} \approx \mathbf{0}$) by a linear function of \mathbf{d} , and $x + \bar{\mathbf{v}}^{p'}$ to be approximated locally (for $p' \approx p$) by $x + \bar{\mathbf{v}}^p + \overline{\mathfrak{D}^*(x, p, (p \rightarrow p')_p, \mathbf{v})}^p$, so that we have a “smooth” action field. There would seem to be enough similarities between the finitary and the general cases to make a generalization possible, yet also enough differences to make that generalization nontrivial.

APPENDIX A. MALCEV OPERATIONS AND RELATED SPACES

A1. Let α be a regular premonoidal action of a vector space V on a non-empty set X (see § 16). We can define a ternary operation

$$\kappa : X \times X \times X \rightarrow X, \quad (x, y, z) \mapsto [x, y, z] := x + \overline{(y \rightarrow z)}.$$

Let us express the postulates that define a *preaffine* space in terms of κ .

As α is a unital action, we have $x + \bar{0} = x$ for all $x \in X$ (3.1), or equivalently $x + \overline{y \rightarrow y} = x$ for all $x, y \in X$. That is, we have the identity

$$(A.1) \quad [x, y, y] = x.$$

Furthermore, α is a transitive action, so we have $x + \overline{x \rightarrow y} = y$ for all $x, y \in X$ (3.3). Thus, we have the identity

$$(A.2) \quad [x, x, y] = y.$$

A ternary operation satisfying (A.1) and (A.2) is called a *Malcev operation* [3].

As α is a free action as well, we have $(x \rightarrow (x + \bar{v})) = \mathbf{v}$ (3.4) for all $x \in X$, so $(x \rightarrow (x + \overline{(y \rightarrow z)})) = (y \rightarrow z)$ for all $x, y, z \in X$, so $p + \overline{(x \rightarrow (x + \overline{(y \rightarrow z)}))} = p + \overline{(y \rightarrow z)}$ for all $p, x, y, z \in X$. In terms of κ , this is the identity

$$(A.3) \quad [p, x, [x, y, z]] = [p, y, z].$$

Finally, as α is a closed action of sets, we have $\overline{(x \rightarrow y)} + \overline{(y \rightarrow z)} = \overline{(x \rightarrow z)}$ (3.7) for all $x, y, z \in X$. Hence, $(p + \overline{(x \rightarrow y)}) + \overline{(y \rightarrow z)} = p + \overline{(x \rightarrow z)}$ for all $p, x, y, z \in X$. This translates to the identity

$$(A.4) \quad [[p, x, y], y, z] = [p, x, z].$$

In a *semipreaffine* space (A.1), (A.2), and (A.3) hold; a *strictly* semipreaffine space is one where (A.4) does not hold.

κ is said to be *commutative* when $[x, y, z] = [z, y, x]$ for all $x, y, z \in X$, and *associative* when $[[x, y, z], r, t] = [x, y, [z, r, t]]$ for all $x, y, z, r, t \in X$. A space with an associative Malcev operation has many names, including *heap* and *groud*. A Malcev operation satisfying either (A.3) or (A.4) is sometimes said to be *semiassociative* [3]. Note that if κ is associative then (A.1) implies (A.3) and (A.2) implies (A.4). Conversely, if (A.3) and (A.4) hold then

$$[[x, y, z], r, t] = [[x, y, z], z, [z, r, t]] = [x, y, [z, r, t]],$$

so a Malcev operation is associative if and only if (A.3) and (A.4) hold.

It is easy to verify that a preaffine space is torsion-free (as described in § 23 and § 28) if and only if $[y, x, z] = [z, x, y]$ for all $x, y, z \in X$, and that a semipreaffine space is flat (as described in § 25 and § 29) if and only if $[[p, x, y], y, z] = [p, x, z]$ for all $p, x, y, z \in X$, that is, if (A.4) holds so that X is in fact a preaffine space. Thus, a commutative κ gives a torsion-free space, while an associative κ gives a curvature-free space. In other words, we can have non-zero curvature only in a strictly semipreaffine space (or a corresponding strictly multiaffine space).

A2. Kock [4] defines an *affine connection* on X as a ternary operation λ on X such that $\lambda(x, x, y) = y$ and $\lambda(x, y, x) = y$ for all $x, y \in X$. (We disregard here the “infinitesimal” part of Kock’s argument.) Writing $\lambda(x, y, z)$ as $[y, x, z]$, these identities are rendered as $[x, x, y] = y$ (A.2) and $[y, x, x] = y$ (A.1), so λ is in effect a Malcev operation. Kock then proves (using further assumptions) that $\lambda(y, x, \lambda(x, y, z)) = z$ and $\lambda(z, \lambda(x, y, z), x) = y$, or equivalently

$$(A.5) \quad [x, y, [y, x, z]] = z$$

and

$$(A.6) \quad [[y, x, z], z, x] = y.$$

It is clear that (A.5) is implied by (A.2) and (A.3) while (A.6) is implied by (A.1) and (A.4). On the other hand, a “Kock space” is not necessarily a preaffine space. As we shall see, it is sometimes possible to replace (A.3) by (A.5) and (A.4) by (A.6) without losing desired properties of the spaces considered.

A3. In Section 3, we defined $x + \overline{(y \rightarrow z)}$ in terms of an action of a vector space on X . One can think of $[x, y, z]$ as just a notational shorthand for $x + \overline{(y \rightarrow z)}$. On the other hand, one can regard κ , required to satisfy (A.1), (A.2) and one or more of (A.3)–(A.6), as a Malcev operation that makes it possible to define preaffine and other spaces *intrinsically*, without reference to an action of an external vector space. Thus, a preaffine space is given by an abstract ternary operator satisfying (A.1)–(A.4), a semipreaffine space is given by an abstract ternary operator satisfying (A.1)–(A.3), and so on.

In terms of κ , a *translation* on X is a function

$$(A.7) \quad [-, a, b] : X \rightarrow X, \quad x \mapsto [x, a, b] \quad a, b \in X.$$

The set of all translations of this form is denoted \mathcal{T}_X . As $[a, a, b] = b$ for all $a, b \in X$ by (A.2), we have $[-, a, b](a) = b$. For any $a \in X$, the translation $[-, a, a]$ is the identity map ϵ_X on X because $[x, a, a] = x$ for any $x, a \in X$ by (A.1). Conversely, if a transformation of the form $[-, a, b]$ is ϵ_X so that $[-, a, b](a) = a$ then $b = [a, a, b] = a$ by (A.2).

Note that $[-, a, b] = [-, a', [a', a, b]]$ by (A.3). Conversely if $[-, a', b'] = [-, a, b]$ so that $[a', a', b'] = [a', a, b]$ then $b' = [a', a, b]$ by (A.2). Thus,

$$[-, a', b'] = [-, a, b] \iff b' = [a', a, b].$$

Also note that if $[-, a, b](p) = p$ for some $p \in X$ so that $[p, a, b] = p$ then

$$[-, a, b](x) = [x, a, b] = [x, p, [p, a, b]] = (x, p, p) = x$$

for any $x \in X$ by (A.3) and (A.1), meaning that if $[-, a, b]$ has a fixed point then $[-, a, b] = \epsilon_X$. Thus translations defined in terms of κ has this property in common with translations defined in terms of group actions (see § 9). In addition, if (A.6) holds then $[-, a, b] \circ [-, b, a] = \epsilon_X$, because

$$[-, a, b]([-, b, a](x)) = [[x, b, a], a, b] = x,$$

so a translation as defined in (A.7) is a bijective function.

By definition, $[-, c, d] \circ [-, a, b](x) = [-, c, d]([-, a, b](x)) = [[x, a, b], c, d]$. Thus, the composite, called the *sum*, of $[-, a, b]$ and $[-, c, d]$ is the transformation

$$(A.8) \quad \begin{aligned} &[-, a, b] + [-, c, d] : X \rightarrow X \\ &(x \mapsto [[x, a, b], c, d]) =: [-, a, b], c, d]. \end{aligned}$$

This definition is legitimate when (A.2) and (A.3) hold, because then $[-, a', b'] = [-, a, b]$ and $[-, c', d'] = [-, c, d]$ implies

$$[[x, a', b'], c', d'] = [[x, a', [a', a, b]], c', [c', c, d]] = [[x, a, b], c, d].$$

To extend the definition of addition of translations, we set

$$\begin{aligned} [-, a_1, b_1] + ([-, a_2, b_2] + [-, a_3, b_3]) &:= ([-, a_2, b_2] + [-, a_3, b_3]) \circ [-, a_1, b_1]; \\ ([-, a_1, b_1] + [-, a_2, b_2]) + [-, a_3, b_3] &:= [-, a_3, b_3] \circ ([-, a_1, b_1] + [-, a_2, b_2]). \end{aligned}$$

Then

$$\begin{aligned} [-, a_1, b_1] + ([-, a_2, b_2] + [-, a_3, b_3]) &= ([-, a_3, b_3] \circ [-, a_2, b_2]) \circ [-, a_1, b_1] \\ &= [-, a_3, b_3] \circ ([-, a_2, b_2] \circ [-, a_1, b_1]) = ([-, a_1, b_1] + [-, a_2, b_2]) + [-, a_3, b_3]. \end{aligned}$$

We may thus omit parentheses from sums involving translations, since all ways of completing an expression of the form

$$[-, a_1, b_1] + [-, a_2, b_2] + \dots + [-, a_n, b_n]$$

with parentheses yield the same transformation. We call a transformation of this kind an *iteration of translations*, and denote the set of all iterations of translations on X by \mathcal{T}_X^* . Translations are thus a special case of iterations of translations, and addition of translations is a special case of addition of iterations of translations. Note that

$$\begin{aligned} &[-, a_1, b_1] + [-, a_2, b_2] + \dots + [-, a_n, b_n] \\ &= [-, a_n, b_n] \circ \dots \circ [-, a_2, b_2] \circ [-, a_1, b_1] \\ &= [[[-, a_1, b_1], a_2, b_2], \dots], a_n, b_n], \end{aligned}$$

so the sum of iterations of translations again does not depend on the choice of pairs of points representing the translations involved.

It is clear that $(\mathcal{T}_X^*, +)$, where $+$ denotes addition of iterations of translations, is an associative magma.

By (A.1), $[-, e, e](x) = [x, e, e] = x$ for any $x, e \in X$, so $[-, e, e] = \epsilon_X$ for all $e \in X$. By (A.1), we also have

$$\begin{aligned} ([-, e, e] + [-, c, d])(x) &= [[-, e, e], c, d](x) = [[x, e, e], c, d] = [x, c, d] = [-, c, d](x); \\ ([-, a, b] + [-, e, e])(x) &= [[-, a, b], e, e](x) = [[x, a, b], e, e] = [x, a, b] = [-, a, b](x). \end{aligned}$$

Thus, for any $e \in X$ we have

$$\begin{aligned} [-, a, b] + [-, e, e] &= [-, e, e] + [-, a, b] = [-, a, b]; \\ [-, a_1, b_1] + \dots + [-, a_n, b_n] + [-, e, e] &= [-, e, e] + [-, a_1, b_1] + \dots + [-, a_n, b_n] \\ &= [-, a_1, b_1] + \dots + [-, a_n, b_n] \end{aligned}$$

for any $[-, a, b] \in \mathcal{T}_X$ and $[-, a_1, b_1] + \dots + [-, a_n, b_n] \in \mathcal{T}_X^*$. Hence, $(\mathcal{T}_X^*, +)$ is a monoid, since it is an associative magma.

Furthermore, if (A.6) holds then

$$([-, a, b] + [-, b, a])(x) = [[-, a, b], b, a](x) = [[x, a, b], b, a] = x,$$

so

$$[-, a, b] + [-, b, a] = \epsilon_X = [-, e, e].$$

Thus, every iteration of translations $[-, a_1, b_1] + \dots + [-, a_n, b_n]$ has an inverse iteration of translations $[-, b_n, a_n] + \dots + [-, b_1, a_1]$, so $(\mathcal{T}_X^*, +)$ is a group since it is a monoid. In particular, if κ is associative then $(\mathcal{T}_X^*, +)$ is a group.

If κ is associative, we also have

$$(A.9) \quad [-, a, b] + [-, c, d] = (x \mapsto [[x, a, b], c, d]) = (x \mapsto [x, a, [b, c, d]]).$$

Hence, $[-, a, b] + [-, c, d]$ is a translation as well, and $(\mathcal{T}_X, +)$ is a group because $(\mathcal{T}_X^*, +)$ is a group (compare § 9).

A4. Fix some $e \in X$. We call a translation that can be written in the form $[-, e, b]$ a *pointed translation* relative to e . Any translation can be written as a pointed translation in at most one way since

$$[-, e, b] = [-, e, b'] \iff b = b'$$

by (A.2). Hence, we can define the *pointed sum* of the pointed translations $[-, e, b]$ and $[-, e, d]$ as the pointed translation

$$(A.10) \quad \begin{aligned} & [-, e, b] +_e [-, e, d] : X \rightarrow X, \\ & (x \mapsto [x, e, [b, e, d]]) =: [-, e, [b, e, d]]. \end{aligned}$$

Note that $+_e$ is not in general an associative operation, since

$$\begin{aligned} & ([-, e, b] +_e [-, e, d]) +_e [-, e, f] = [-, e, [[b, e, d], e, f]]; \\ & [-, e, b] +_e ([-, e, d] +_e [-, e, f]) = [-, e, [b, e, [d, e, f]]], \end{aligned}$$

but if κ is associative then $+_e$ is obviously associative as well.

If κ is commutative then $[x, e, [b, e, d]] = [x, e, [d, e, b]]$, so

$$[-, e, b] +_e [-, e, d] = [-, e, d] +_e [-, e, b],$$

so the binary operation $+_e$ is commutative as well.

As $[x, e, [b, e, e]] = [x, e, b]$ by (A.1) and $[x, e, [e, e, b]] = [x, e, b]$ by (A.2), we have

$$(A.11) \quad [-, e, b] +_e [-, e, e] = [-, e, e] +_e [-, e, b] = [-, e, b]$$

for any $e, b \in X$, so $[-, e, e]$ is the unique identity under pointed addition of translations.

Also, if (A.5) holds then $[x, e, [b, e, [e, b, e]]] = [x, e, e]$, so $[-, e, [e, b, e]]$ is a right inverse of $[-, e, b]$, and if (A.6) holds then $[x, e, [[e, b, e], e, b]] = [x, e, e]$, so $[-, e, [e, b, e]]$ is a left inverse of $[-, e, b]$. Thus, if (A.5) and (A.6) hold then

$$(A.12) \quad [-, e, b] +_e [-, e, [e, b, e]] = [-, e, [e, b, e]] +_e [-, e, b] = [-, e, e].$$

In particular, if κ is associative then $[-, e, [e, b, e]]$ is the unique inverse of $[-, e, b]$ under pointed addition of translations relative to e .

Recall that if (A.2) and (A.3) hold then $[-, a, b] = [-, e, b']$ if and only if $b' = [e, a, b]$. This means that every translation $[-, a, b]$ is a pointed translation which can be written in the form $[-, e, b']$ in exactly one way, namely as $[-, e, [e, a, b]]$.

(A.10) can then be written as

$$\begin{aligned} & [-, a, b] +_e [-, c, d] : X \rightarrow X, \\ & (x \mapsto [x, e, [[e, a, b], e, [e, c, d]]]) = [-, e, [[e, a, b], e, [e, c, d]]]. \end{aligned}$$

This implies that (A.12) can be rendered as

$$[-, a, b] +_e [-, b, a] = [-, b, a] +_e [-, a, b] = [-, e, e].$$

Note that if (A.3) and (A.4) hold then

$$\begin{aligned}
& ([-, a, b] +_e [-, c, d]) (x) = ([-, a, b] +_e [-, b, [b, c, d]]) (x) \\
& = [x, e, [[e, a, b], e, [e, b, [b, c, d]]]] = [x, e, [[e, a, b], b, [b, c, d]]] \\
& = [x, e, [e, a, [b, c, d]]] = [x, a, [b, c, d]] = [[x, a, b], b, [b, c, d]] \\
& = [[x, a, b], c, d] = ([-, a, b] + [-, c, d]) (x).
\end{aligned}$$

That is, if κ is an associative Malcev operation then

$$[-, a, b] +_e [-, c, d] = [-, a, b] + [-, c, d]$$

for any $e \in X$.

A5. There is another, more well-known way of recovering a binary operation from κ satisfying (A.1)–(A.4), this time primarily involving individual points. For any fixed $e \in X$ the *pointed combination* of x and y is the point $x \diamond_e y$ defined by

$$X \times X \rightarrow X, \quad (x, y) \mapsto x \diamond_e y := [x, e, y].$$

If κ is associative then $[[x, e, y], e, z] = [x, e, [y, e, z]]$, so $(x \diamond_e y) \diamond_e z = x \diamond_e (y \diamond_e z)$.

In general, $[x, e, e] = x$ for any $x \in X$ by (A.1) so that e is a right identity element, and $[e, e, x] = x$ for any $x \in X$ by (A.2) so that e is a left identity element. Hence, if κ is a Malcev operation then e is the unique identity in (X, \diamond_e) . Also, if (A.5) holds then $[x, e, [e, x, e]] = e$ so that $[e, x, e]$ is a right inverse of x , and if (A.6), holds then $[[e, x, e], e, x] = e$, so that $[e, x, e]$ is a left inverse of x . In particular, if κ is associative then $[e, x, e]$ is the unique inverse of x .

This means that if κ is an associative Malcev operation then (X, \diamond_e) is a group. If κ is commutative then (X, \diamond_e) is evidently commutative.

Note that for any choice of identity element e in X there is a bijection

$$\phi : X \rightarrow \mathcal{T}_X, \quad p \mapsto [-, e, p].$$

Recall that $[-, e, p] +_e [-, e, q] = [-, e, [p, e, q]]$. Thus, $\phi(p \diamond_e q) = [-, e, [p, e, q]] = [-, e, p] +_e [-, e, q] = \phi(p) +_e \phi(q)$.

Also, $\phi(e) = [-, e, e]$ so that $\phi(e) +_e [-, e, p] = [-, e, [e, e, p]] = [-, e, p]$ by (A.2) and $[-, e, p] +_e \phi(e) = [-, e, [p, e, e]] = [-, e, p]$ by (A.1).

If κ is an associative Malcev operation then

$$\begin{aligned}
& \phi(p^{-1})(x) = [x, e, [e, p, e]] = [x, p, e] = [-, p, e](x); \\
& [-, e, p] +_e \phi(p^{-1}) = [-, e, p] + \phi(p^{-1}) = [[-, e, p], p, e] = [-, e, e]; \\
& \phi(p^{-1}) +_e [-, e, p] = \phi(p^{-1}) + [-, e, p] = [[-, p, e], e, p] = [-, p, p] = [-, e, e].
\end{aligned}$$

Thus, if κ is associative then ϕ is a canonical isomorphism between (X, \diamond_e) and $(\mathcal{T}_X, +_e)$.

Furthermore, if κ is associative there is a canonical isomorphism

$$\psi : (X, \diamond_e) \rightarrow (X, \diamond_{e'}), \quad x \mapsto x \diamond_e e'$$

for any $e, e' \in X$. In fact,

$$\psi(x \diamond_e y) = (x \diamond_e y) \diamond_e e' = \psi(x) \diamond_{e'} \psi(y)$$

because

$$[[x, e, y], e, e'] = [x, e, [y, e, e']] = [[x, e, e'], e', [y, e, e']].$$

Thus, from any associative Malcev operation on X we can recover a group (X, \diamond) , unique up to a canonical isomorphism. This group can be interpreted as a group acting on X by a group action of the form $p \mapsto (-, e, p)$.

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